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DUAL RECIPROCITY BOUNDARY ELEMENT METHOD FOR THE TIME-DEPENDENT STOKES FLOWS

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ABSTRACT

This paper presents a boundary element formulation for the transient Stokes equations in which the well known closed form fundamental solution to the steady Stokes equations is employed and the time derivative is taken to the boundary with dual reciprocity method. This approach has the advantage of simplicity of formulation and implementation in relation to the alternative boundary element schemes previously presented. In addition in this paper the dual reciprocity method is presented in a more formal mathematical way using well established interpolation theories which guarantee the convergence of the method. Results are presented for a series of three-dimensional internal problems in which the accuracy of the method is shown.

KEY WORDS Dual reciprocity method Transient Stokes equations 3-D pipe flow

INTRODUCTION

The boundary element method is now a well established technique for the analysis of engineering problems. One of its main advantages is the considerable reduction in data preparation, in relation to domain methods, as only surface elements are necessary. The basis of the method is that a fundamental solution is used to take some or all of the terms in the governing equation to the boundary.

Although theoretically any linear partial differential equation has a fundamental solution, in some cases this solution cannot be expressed in a closed form, requiring numerical integration in order to be evaluated. The use of such fundamental solutions for practical programming purposes is not very convenient. If the problem can be expressed in such a way that a simpler fundamental solution with a closed form may be used and some terms expressed as domain integrals, considerable computational advantage may be obtained.

In early boundary element analysis the evaluation of domain integrals was done using cell integration, a technique, which, whilst effective and general, made the method lose its boundary only nature introducing an additional internal discretization.

Several methods have been developed to take domain integrals to the boundary in order to eliminate the need for internal cells, one of the most effective to date being the dual reciprocity method (DRM). This method was introduced by Nardini and Brebbia[1]; the method is general and straightforward to apply.

In this paper the boundary element method will be applied to non-permanent Stokes system

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of equations:

$$\rho \frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{for all } x \in \Omega$$
$$\frac{\partial u_i}{\partial x_i} = 0$$

here, $\vec{u}$ is the flow velocity, ρ is the fluid density, p is the pressure, μ is the dynamic viscosity, and Ω is a bounded three-dimensional domain, with a surface S as contour, that we will consider to be of Lyapunov type.

The velocity field satisfies the non-slip boundary condition on the surface S, i.e. $u_i(\xi) = U_i(\xi)$ for all $\xi \in S$.

The above system of equations is a first approximation to the complete Navier–Stokes system of equations for incompressible flow, when the flow phenomena occur over a time scale of the order $O(L^2/\nu)$, where L is a characteristic length and $\nu = \mu/\rho$ is the kinematic viscosity, and the Reynolds' number, $R_e = UL/\nu$, is small, then the acceleration term $\partial u_i/\partial t$ is of the same order as the viscous terms, but the non-linear term will be $O(R_e)$ smaller than the remaining terms.

This problem can be analysed using the following integral equation approaches.

The fundamental solution to the full equations, $(\vec{U}^k, P^k)$, given by Ladyzhenskaya[2], may be employed:

$$U_i^k(x, y, t) = B(x, y, t)\delta_{ik} + \frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{1}{2\pi^{3/2} r}\int_0^{r/2\sqrt{\nu t}} \exp(-\tau^2)\, d\tau\right)$$
$$P^k(x, y, t) = -\frac{\partial}{\partial x_k}\frac{\delta(t)}{4\pi r}$$

where $B(x, y, t) = (4\pi\nu t)^{-3/2} \exp(-r^2/4\nu t)$ is the fundamental solution of the heat equation, δ_{ik} is the Kronecker delta, $\delta(t)$ is the Dirac delta function and $r = |x - y|$.

This approach leads to a time-boundary integral equation, see for instance Gavze[3]. The principal difficulty encountered is that the above fundamental solution cannot be evaluated explicitly, requiring the use of numerical integration because of the integral seen in the first expression; on the other hand, the use of this formulation produces a boundary integral expression for the complete problem without domain terms. Another possibility is to transform the non-permanent Stokes system of equations to the frequency domain as done for instance by Pozrikidis[4]. For the transformed equation, a simple closed form fundamental solution exists, once again the resulting boundary integral equation does not contain domain terms. The main disadvantage of this approach is that it is necessary to carry out the inverse transform on the solution.

An alternative approach, to be employed here, is to use the fundamental solution to the permanent Stokes equations, and express the time derivative as a domain integral. This fundamental solution has a simple closed form, however the domain integral will introduce extra numerical computations.

A convenient method for taking the domain integral to the boundary is the dual reciprocity method. In this way a boundary integral equation will be obtained without domain terms, using a closed form fundamental solution and without the need to recourse to inverse transforms. This approach has been suggested by Jin and Brown[5] for the two-dimensional case; however, no numerical implementation was carried out.

DUAL RECIPROCITY METHOD

Consider the unsteady Stokes equation written in a non-homogeneous steady form:

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i} = g_i(x) \qquad \frac{\partial u_i}{\partial x_i} = 0 \quad \text{with } g_i = \rho \frac{\partial u_i}{\partial t} \tag{1a,b,c}$$

the integral representation or Green's formulae, for this case for a point $x \in \Omega$ are given by Ladyzhenskaya[2] as:

$$u_k(x) = \int_S K_{kj}(x, y) u_j(y)\, dS_y - \int_S U_i^k(x, y) \sigma_{ij}(\vec{u}(y), p(y)) n_j(y)\, dS_y + \int_\Omega U_i^k(x, y) g_i(y)\, dy \tag{2}$$

where $\sigma_{ij}(\vec{u}, p) = -p\delta_{ij} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the stress tensor corresponding to the flow field $(\vec{u}, p)$, $\vec{U}^k$ is the velocity field of the fundamental solution of the Stokes' equation known as a Stokeslet, with a corresponding pressure q^k:

$$U_i^k(x, y) = -\frac{1}{8\pi\mu}\left(\frac{\delta_{ik}}{r} + \frac{(x_i - y_i)(x_k - y_k)}{r^3}\right) \tag{3a}$$

$$q^k(x, y) = -\frac{(x_k - y_k)}{4\pi r^3} \tag{3b}$$

and

$$K_{ij}(x, y) = \sigma_{ij}(\vec{U}^k, -q^k) n_k(y) = -\frac{3}{4\pi} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5} n_k(y) \tag{3c}$$

In order to express the boundary integral in (2) in terms of equivalent domain integrals, a dual reciprocity approximation is introduced (see Partridge *et al.*[6]). The basic idea is to expand the time-derivative $\partial u_i/\partial t$ in the form,

$$g_i(x) = \rho \frac{\partial u_i}{\partial t} = \sum_{m=1}^{p} f(x, y^m) \alpha_l^m(t) \delta_{il} \tag{4}$$

The above series involves a set of known functions $f(x, y^m)$ which are dependent only on geometry, a set of unknown vector coefficients $\vec{\alpha}^m$ which are time-dependent only, and y^m, with $m = 1, 2, \ldots, p$, been p fixed collocation points chosen to be in the closed domain where the function is approximated (also called nodal points). With this approximation, the domain integral becomes:

$$\int_\Omega U_i^k(x, y) g_i(y)\, dy = \sum_{m=1}^{p} \alpha_l^m \int_\Omega U_i^k(x, y) f(y, z^m) \delta_{il}\, dy \tag{5}$$

To reduce the last domain integral in (5) to an equivalent boundary integral, let us define a new auxiliary non-homogeneous Stokes' flow field $(\hat{U}_i^l(x, y^m)e_i, \hat{p}^l(x, y^m))$, for each collocation point, y^m, in the following way:

$$\mu \frac{\partial^2 \hat{U}_i^l(x, y^m)}{\partial x_j \partial x_j} - \frac{\partial \hat{p}^l(x, y^m)}{\partial x_i} = f(x, y^m) \delta_{il} \tag{6a}$$

$$\frac{\partial \hat{U}_i^l}{\partial x_i} = 0 \tag{6b}$$

Applying Green's formulae to the non-homogeneous Stokes' flow field $(\hat{U}_i^l(x, y^m)e_i, \hat{p}^l(x, y^m))$, we obtain:

$$\hat{U}_k^l(x,z^m) = \int_S K_{kj}(x,y)\hat{U}_j^l(y,z^m)\,\mathrm{d}S_y - \int_S U_i^k(x,y)\sigma_{ij}(\hat{U}_i^l(y,z^m)e_i, \hat{p}^l(y,z^m))n_j(y)\,\mathrm{d}S_y + \int_\Omega U_i^k(x,y)f(y,z^m)\delta_{il}\,\mathrm{d}y \tag{7}$$

or equivalently:

$$\int_\Omega U_i^k(x,y)f(y,z^m)\delta_{il}\,\mathrm{d}y = \hat{U}_k^l(x,z^m) - \int_S K_{kj}(x,y)\hat{U}_j^l(y,z^m)\,\mathrm{d}S_y + \int_S U_i^k(x,y)\sigma_{ij}(\hat{U}_i^l(y,z^m)e_i, \hat{p}^l(y,z^m))n_j(y)\,\mathrm{d}S_y \tag{7a}$$

Substituting the last equation into (5), the domain integral can be recast in the form:

$$\int_\Omega U_i^k(x,y)g_i(y)\,\mathrm{d}y = \sum_{m=1}^{p}\alpha_l^m\left\{\hat{U}_k^l(x,z^m) - \int_S K_{kj}(x,y)\hat{U}_j^l(y,z^m)\,\mathrm{d}S_y + \int_S U_i^k(x,y)\sigma_{ij}(\hat{U}_i^l(y,z^m)e_i, \hat{p}^l(y,z^m))n_j(y)\,\mathrm{d}S_y\right\} \tag{8}$$

and using the resulting expression in (2), one finally arrives at a boundary only integral representation formula for the velocity field at point $x\in\Omega$, in the form:

$$u_k(x) = \int_S K_{kj}(x,y)u_j(y)\,\mathrm{d}S_y - \int_S U_i^k(x,y)\sigma_{ij}(\vec{u}(y),p(y))n_j(y)\,\mathrm{d}S_y + \sum_{m=1}^{p}\left\{\hat{U}_k^l\,\alpha_l^m(x,z^m) - \int_S K_{kj}(x,y)\hat{U}_j^l(y,z^m)\,\mathrm{d}S_y + \int_S U_i^k(x,y)\sigma_{ij}(\hat{U}_i^l(y,z^m)e_i, \hat{p}^l(y,z^m))n_j(y)\,\mathrm{d}S_y\right\} \quad x\in\Omega \tag{9}$$

Applying the Dirichlet boundary condition at points $\xi\in S$, i.e. $u_i(\xi) = U_i(\xi)$, to the velocity field given by the integral representation formula (9), and using the known jump property of the double layer potential, we obtain the following system of Fredholm's integral equations of the first kind for the unknown local stress forces, $\sigma_{ij}(\vec{u}(y),p(y))n_j(y) = \phi_i(y)$, in terms of boundary only integrals:

$$\int_S U_i^k(\xi,y)\phi_i(y)\,\mathrm{d}S_y = -U_k(\xi) + c_{kj}(\xi)U_j(\xi) + \int_S K_{kj}(\xi,y)U_j(y)\,\mathrm{d}S_y + \sum_{m=1}^{p}\alpha_l^m\left\{\hat{U}_k^l(\xi,z^m) - c_{kj}(\xi)\hat{U}_j^l(\xi,z^m) - \int_S K_{kj}(\xi,y)\hat{U}_j^l(y,z^m)\,\mathrm{d}S_y + \int_S U_i^k(\xi,y)\hat{\phi}_i^l(y,z^m)\,\mathrm{d}S_y\right\} \quad \xi\in S \tag{10}$$

where $\sigma_{ij}(\hat{U}_i^l(y,z^m)e_i, \hat{p}^l(y,z^m))n_j(y) = \hat{\phi}_i^l(y,z^m)$, $c_{ij}(\xi) = \Theta(\xi)\delta_{ij}$, and $\Theta(\xi)$ is the solid angle at the point $\xi\in S$.

CHOICE OF FUNCTION(f)

Previous work on dual reciprocity has shown[6] that although a variety of functions can in principle be used as a basic approximation function, best results are normally obtained with simple expansions, the most popular of which is $f = 1 + R$, where R is the distance between pre-specified fixed collocation points, y^m, and a field point x where the function is approximated, i.e. $R = |x - y^m|$. In the reference given, as well as in previous DRM literature, the choice is based on experience rather than formal mathematical analysis. However, recent mathematical work, related to the theory of mathematical interpolation, and unrelated to the integral equations literature, based on the so-called radial functions, has partly sustained these numerical findings, the approximation function, $f = 1 + R$, defined previously, in general use in DRM is just one such radial function. This mathematical work provides general convergence criteria for the interpolation series based on radial functions. The convergence is derived from the fact that the behaviour of these interpolation series is local, i.e. increasing the number of fixed nodal points makes the interpolation more localized. It is interesting to note that although the basic functions are defined globally, the interpolation series have local behaviour. For the case of radial functions defined by the above f function in one-dimensional space, these new theories have shown that the resulting interpolation series happens to be a natural spline interpolation. Much of this mathematical work has addressed the properties of these approximations when the data points, collocation points y^m, form an infinite regular grid, because this structure allows the order of accuracy to be derived when an interpolation or quasi-interpolation procedure is applied to a smooth underlying function. Unfortunately, the subject of non-regular grids has insufficient theoretical support, and is an on-going topic of mathematical research. In practical application of the DRM, the grid for the interpolation series is usually non-regular (for more detail on the theory of radial basis function approximation, see Powell[7,8]).

By evaluating (4) at all nodal points and inverting, one arrives at:

$$\alpha_l^m = (F_{im}^l)^{-1} g_i(y^m) = (F_{im}^l)^{-1} \rho \frac{\partial u_i(y^m)}{\partial t} \tag{11}$$

Micchelli[9] has proved that for the case when the nodal points are all distinct, the matrix $(\mathbf{F}^l)^{-1}$, resulting from the basic function defined previously, for all positive integers p, (the dimension of the matrix), and n, (the dimension of the space, R^n), is always non-singular, i.e. the matrix is invertible. Therefore, as long as the function $\vec{g}(x)$ is regular, then the above vector coefficients $\vec{\alpha}^m$, that are used in the dual reciprocity schemes, are well defined.

In order to find the corresponding particular solution of the non-homogeneous Stokes system of (6a,b), we will use an approach suggested by Happel and Brenner[10] to find the fundamental solution of the Stokes homogeneous equations, defining the second rank tensor $\hat{U}_i^l(x, y^m)$ in terms of an auxiliary potential ψ, as follows:

$$\hat{U}_i^l(x, y^m) = \frac{\partial^2 \psi(R)}{\partial x_k \partial x_k} \delta_{il} - \frac{\partial^2 \psi(R)}{\partial x_i \partial x_l}, \qquad R = |x - y^m| \tag{12}$$

By differentiating the above equation, we find that, for any choice of the potential ψ, the continuity equation (6b) is automatically satisfied. Substituting (12) into (6a), we obtain:

$$\mu\left(\frac{\partial^4 \psi(R)}{\partial x_k \partial x_k \partial x_k \partial x_k} \delta_{il} - \frac{\partial^4 \psi(R)}{\partial x_k \partial x_k \partial x_i \partial x_l}\right) - \frac{\partial \hat{p}^l}{\partial x_i} = (1 + R)\delta_{il} \tag{13}$$

If we now assume that ψ satisfies the non-homogeneous biharmonic equation:

$$\mu \frac{\partial^4 \psi(R)}{\partial x_k \partial x_k \partial x_k \partial x_k} = (1 + R) \tag{14}$$

the solution of which is given by $\mu\psi = R^4/120 + R^5/360$, then, the pressure field has to satisfy the following equation:

$$\mu \frac{\partial^4 \psi(R)}{\partial x_k \partial x_k \partial x_i \partial x_l} + \frac{\partial \hat{p}^l}{\partial x_i} = 0 \tag{15}$$

or

$$\hat{p}^l(x, y^m) = -\mu \frac{\partial^3 \psi(R)}{\partial x_k \partial x_k \partial x_l} \tag{16}$$

The biharmonic potential ψ, thus obtained is substituted into (12) and (16), in order to obtain the following expression for the non-homogeneous Stokes flow field $(\hat{U}_i^l(x, y^m)e_i, \hat{P}^l(x, y^m))$

$$\hat{U}_i^l(x, y^m) = \left[R^2\left(\frac{4}{30} + \frac{5R}{72}\right)\delta_{il} - (x_i - y_i^m)(x_l - y_l^m)\left(\frac{1}{15} + \frac{R}{24}\right)\right] \tag{17a}$$

and

$$\hat{p}^l(x, y^m) = -(x_l - y_l^m)\left(\frac{1}{3} + \frac{R}{4}\right) \tag{17b}$$

with the stress tensor $\sigma_{ij}(\hat{U}_i^l(y, z^m)e_i, \hat{p}^l(y, z^m))$ given by:

$$\sigma_{ij}(\hat{U}_i^l(y, z^m)e_i, \hat{p}^l(y, z^m)) = \left[(x_l - y_l^m)\left(\frac{1}{5} + \frac{R}{6}\right)\delta_{ij} + (x_i - y_i^m)\left(\frac{1}{5} + \frac{R}{6}\right)\delta_{lj} + \right.$$
$$\left. (x_j - y_j^m)\left(\frac{1}{5} + \frac{R}{6}\right)\delta_{il} - \frac{(x_l - y_l^m)(x_i - y_i^m)(x_j - y_j^m)}{12R}\right] \tag{18}$$

NUMERICAL SOLUTION

For the numerical solution of the problem, (10) is written in a discretized form in which the boundary integrals are approximated by a summation of integrals over individual boundary elements, i.e.

$$\sum_{n=1}^{N} \int_{\Delta S_n} U_i^k(\xi, y)\phi_i(y)\,dS_y = -U_k(\xi) + c_{kj}(\xi)U_j(\xi) + \sum_{n=1}^{N} \int_{\Delta S_n} K_{kj}(\xi, y)U_j(y)\,dS_y +$$
$$\sum_{m=1}^{p} \alpha_l^m \left\{ \hat{U}_k^l(\xi, z^m) - c_{kj}(\xi)\hat{U}_j^l(\xi, z^m) + \right.$$
$$\sum_{n=1}^{N} \int_{\Delta S_n} K_{kj}(\xi, y)\hat{U}_j^l(y, z^m)\,dS_y +$$
$$\left. \sum_{n=1}^{N} \int_{\Delta S_n} U_i^k(\xi, y)\hat{\phi}_i^l(y, z^m)\,dS_y \right\} \quad \xi \in S \tag{19}$$

Equation (19) can be written in matrix form:

$$\mathbf{G}\phi = \mathbf{HU} + \alpha(\mathbf{G}\hat{\phi}^k - \mathbf{HU}^k) \tag{20}$$

In the above system, **G** and **H** are square matrices, the coefficients of which are calculated by integrating products of $\vec{U}^k$ and $\vec{K}^k$ by the interpolation function, respectively, over each boundary element. In the examples considered here, quadratic eight node elements were used.

Details of standard boundary element techniques may be found in, for instance, Brebbia *et al.*[11]. By substituting (11) into (20) we obtain:

$$\mathbf{G}\phi = \mathbf{H}\mathbf{U} + \mathbf{S}\dot{\mathbf{u}} \tag{21}$$

where the dot denotes temporal derivative, and

$$\mathbf{S} = \mathbf{F}^{-1}(\mathbf{G}\hat{\phi}^{k} - \mathbf{H}\hat{\mathbf{U}}^{k}) \tag{22}$$

Equation (21) can be integrated in time using standard finite difference time-stepping techniques. It should be noted that the coefficients of the matrices **H**, **G** and **S** all depend on geometry only, thus they can be computed once only and stored.

For simplicity, a two level time integration scheme will be employed; a linear approximation can be proposed for $\vec{u}$ within each time-step in the form:

$$\begin{aligned} u_u &= (1 - \theta_u)u_i^m + \theta_u u_i^{m+1} \\ \dot{u}_i &= \frac{1}{\Delta t}(u_i^{m+1} - u_i^m) \end{aligned} \tag{23}$$

where θ_u is a parameter which positions the value of $\vec{u}$ between time levels m and $(m+1)$, having value 0.5 in the numerical examples, in such a way that values $\vec{u}_0$ will be the initial conditions for the problem. The unknown vector density $\vec{\phi}$, was taken at the advance time level, $(m+1)$.

Substituting (23) into (21) one obtains:

$$2\mathbf{G}\phi^{m+1} = \left(\mathbf{H} + 2\frac{\mathbf{S}}{\Delta t}\right)\mathbf{U}^{m+1} + \left(\mathbf{H} - 2\frac{\mathbf{S}}{\Delta t}\right)\mathbf{U}^{m} \tag{24}$$

It should be noted that, in (24) the number of nodal points p is equal to the number of boundary nodes, N, plus the number of internal nodes, L. Internal nodes are needed in DRM in order to be able to advance in time, given that the boundary conditions, for the cases to be considered here, are constant in time, having zero initial condition throughout the domain. The formulation is not restricted to this type of problem.

Finally, the above linear algebraic system for the unknown $\vec{\phi}$ at the advanced time-step is solved using a standard Gauss procedure. Letting $\vec{u}_{m+1}$ become $\vec{u}_m$ values for a new time-step can be computed until the desired total time is reached.

Results were obtained for four cases which will be described below, in each case the integral surface was discretized using 64 continuous elements with 8 source points each, in the series (4) we used 277 nodal points, of which 194 were the same source points of the surface elements and the remaining 83 corresponded to internal nodes, and a time-step of $\delta t = 0.01$ was employed.

In the first case, we considered the unsteady flow inside a closed circular cylindrical container, starting from a state of rest, where the top and side walls are rotating with constant angular velocity, and the bottom is fixed. This problem has been analysed by Pao[12] and Bertelà and Gori[13], using a finite difference scheme for low and moderate Reynolds number. The cylinder considered has a unit radius and is of unit length, i.e. a unit aspect ratio. A unit angular velocity was imposed on the top and side walls. In *Tables 1a, b* and *c* we present our results for the circulation at different radial distances, for different times, at three different cross-sections of the pipe, one located a dimensionless distance $L/a = 0.25$ from the moving top, one at the middle of the pipe, and the last at a dimensionless distance $L/a = 0.25$ from the fixed bottom. The results compare well with those given for the low Reynolds number cases in the references cited above. In the Tables, it can be observed that the steady state is reached in a very short time, this fact is due to the motion of the pipe top, as can be concluded from the comparison of the previous results with the case of rotating pipe wall and fixed top and bottom. For this last case, in *Figure 1*, we present our results for the circulation profile, for different times, at a cross-section located at the middle of the pipe.

Table 1a Circulation at different radial distances for different times at a cross-section located a dimensionless distance $L/a = 0.25$ from the moving top

Radial distance	$t = 0.1$	$t = 0.2$	$t = 0.3$
1.00	1.000	1.000	1.000
0.75	0.529	0.543	0.544
0.50	0.217	0.231	0.232
0.25	0.051	0.056	0.056
0.00	0.000	0.000	0.000

Table 1b Circulation at different radial distances for different times at a cross-section located at the middle of the pipe

Radial distance	$t = 0.1$	$t = 0.2$	$t = 0.3$
1.00	1.000	1.000	1.000
0.75	0.484	0.503	0.504
0.50	0.178	0.197	0.199
0.25	0.038	0.046	0.047
0.00	0.000	0.000	0.000

Table 1c Circulation at different radial distances for different times at a cross-section located a dimensionless distance $L/a = 0.25$ from the fixed bottom

Radial distance	$t = 0.1$	$t = 0.2$	$t = 0.3$
1.00	1.000	1.000	1.000
0.75	0.383	0.396	0.398
0.50	0.108	0.122	0.123
0.25	0.021	0.026	0.027
0.00	0.000	0.000	0.000

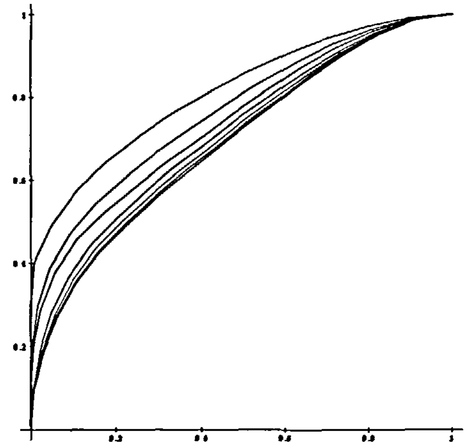

Figure 1 Circulation profile at a cross-section located at the middle of the pipe, for $t = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 4.0$, for the case of rotating pipe wall

In the second case, the unsteady flow in a pipe of aspect ratio 6 was studied, having zero initial velocity field, and imposing a constant parabolic velocity profile, corresponding to a permanent Poiseuille flow with a unit maximum velocity at both ends. *Figure 2* shows the longitudinal velocity profile, at a cross-section at the middle of the pipe, for different times. After a certain dimensionless time ($t \approx 0.7$) the results behave similar to those of the starting flow in a pipe problem[14]. The equivalence between these two problems, after a certain time, can be seen in *Figure 3* where it can be observed that in the vicinity of the middle of the pipe, the centreline longitudinal velocity is practically constant; this corresponds to a constant local pressure gradient, which is the condition for the starting flow in a pipe problem. In *Figure 4*, we present how the centre velocity, at the middle of the pipe, changes with time.

The third case is similar to the previous one, except that a zero velocity profile is imposed at the far end. *Figures 5* and *6* show the longitudinal velocity profile, at a cross-section at the middle of the pipe, and its value at the centreline, for different times, respectively. Finally, in case 4, the same problem as before was studied using a unit uniform velocity profile at the pipe entrance, instead of a parabolic one. *Figures 7* and *8* show the results for the longitudinal velocity profile, at a cross-section at the middle of the pipe, and its value at the centreline, for different times.

It should be pointed out that only the examples dealing with a rotational pipe can be simulated

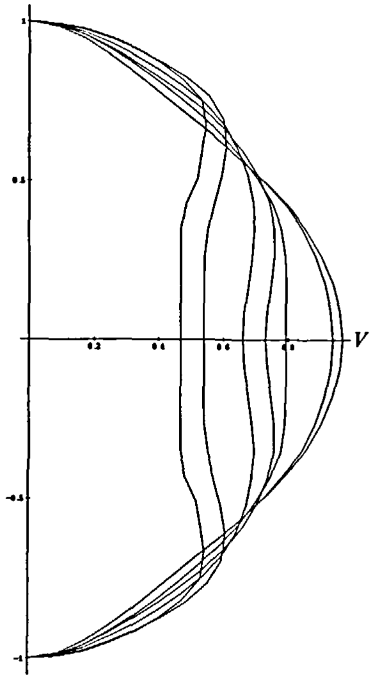

Figure 2 Longitudinal velocity profile at a cross-section located at the middle of the pipe, for $t = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 4.0$, for the second case

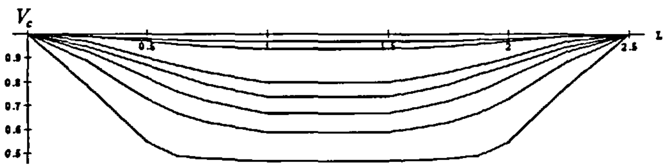

Figure 3 Centreline velocity along the pipe, for $t = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 4.0$, for the second case

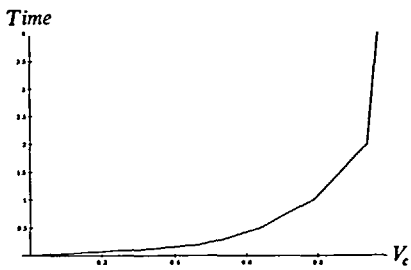

Figure 4 Change of the centre velocity at the middle of the pipe with time, for the second case

physically, the others are numerical examples with some similarities to real flow phenomena; however, they are good to test the numerical model developed here. It is important, also, to point out that for each example, the internal nodes were distributed uniformly inside the pipe. The selection of how many internal nodes and how they are distributed is an on-going research topic in dual reciprocity method.

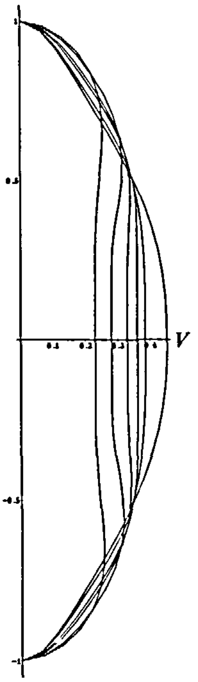

Figure 5 Longitudinal velocity profile at a cross-section located at the middle of the pipe, for $t = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0$, for the third case

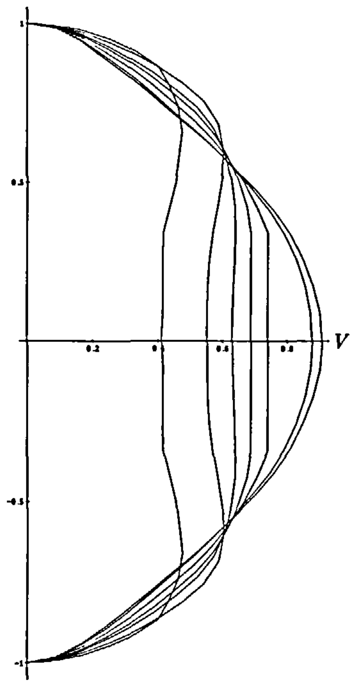

Figure 7 Longitudinal velocity profile at a cross-section located at the middle of the pipe, for $t = 0.2, 0.4, 0.6, 0.8, 1.0, 2.0, 4.0$, for the fourth case

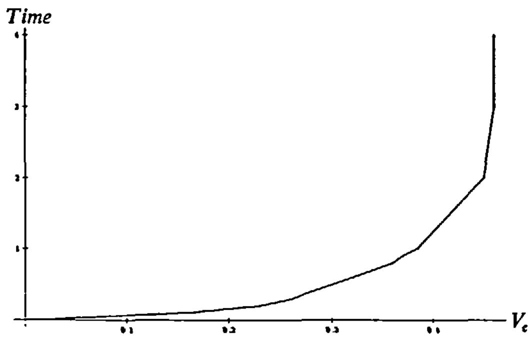

Figure 6 Centreline velocity along the pipe, for different times, for the third case

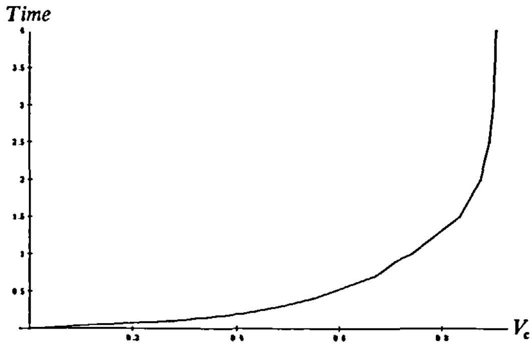

Figure 8 Centreline velocity along the pipe, for different times, for the fourth case

CONCLUSIONS

A dual reciprocity boundary element scheme has been used to solve the unsteady Stokes equations for internal problems. This scheme uses the closed form steady Stokes fundamental solution which is very well known and the numerical integration of which is well established[15]. The scheme has the advantage of simplicity of formulation and implementation in relation to the alternative boundary element approaches, discussed in the Introduction, as numerical integration of the fundamental solution and inverse transforms are not required. The accuracy of the method is shown by the results presented. Further work will now be done implementing the method for exterior problems and for the full Navier–Stokes equations.

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